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Text

Euler–Maclaurin and Gregory interpolants

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Abstract Let a sufficiently smooth function f on $[-1,1]$ be sampled at $n+1$ equispaced points, and let $k \geq 0$ be given. An *Euler–Maclaurin interpolant* to the data is defined, consisting of a sum of a degree k algebraic polynomial and a degree n trigonometric polynomial, which deviates from f by $O(n^{-k})$ and whose integral is equal to the order k Euler–Maclaurin approximation of the integral of f. This interpolant makes use of the same derivatives $f^{(j)}(\pm 1)$ as the Euler–Maclaurin formula. A variant *Gregory interpolant* is also defined, based on finite difference approximations to the derivatives, whose integral (for k odd) is equal to the order k Gregory approximation to the integral.

Keywords Euler–Maclaurin formula · Gregory quadrature

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1 Introduction

It is known that the trapezoidal quadrature rule can be interpreted as an application of a basic scheme for approximating a function: trigonometric interpolation. Given a function $f \in C([-1,1])$ and an integer $n \geq 1$, consider the $(n+1)$-point equispaced grid

$$x_j = -1 + jh, \quad 0 \leq j \leq n, \quad h = 2/n. \tag{1}$$

Let t_n be the unique function of the form

$$t_n(x) = b_1 + b_2 \cos(\pi x) + b_3 \sin(\pi x) + b_4 \cos(2\pi x) + \cdots \quad (n \text{ terms in total}), \tag{2}$$

which we shall call a *degree n trigonometric polynomial,* such that

$$t_n(x_j) = f(x_j), \quad 1 \leq j \leq n-1$$

and

$$t_n(1) = t_n(-1) = \frac{f(1) + f(-1)}{2}.$$

Then

$$\int_{-1}^{1} t_n(x)dx = 2b_1 = T_n(f), \tag{3}$$

where $T_n(f)$ denotes the trapezoidal rule quantity

$$T_n(f) = h \sum_{j=0}^{n}{}' f(jh), \tag{4}$$

an approximation to the integral

$$I(f) = \int_{-1}^{1} f(x)dx. \tag{5}$$

The prime on the summation symbol indicates that the terms $j = 0$ and $j = n$ are multiplied by 1/2. To prove (3) we note that $T_n(f) = T_n(t_n)$, since f and t_n take the same values on the grid, and the trapezoidal rule gives the correct integrals $2b_1, 0, 0, \ldots, 0$ when applied to each of the n terms of (2).

This connection with trigonometric interpolation provides one way to understand the exponential accuracy of the trapezoidal rule in the special case where f is periodic and analytic [22, §7]. And it introduces from the start the paradox that pervades this subject: though a quadrature rule is a local formula, with no connection between the endpoints ± 1, yet it may be helpful in analyzing its behavior to regard ± 1 as a single point of a periodic function.

The aim of this paper is to propose analogous interpolants corresponding to the standard improvements of the trapezoidal rule for nonperiodic function based on endpoint corrections, the *Euler–Maclaurin formula* and *Gregory quadrature.* These quadrature formulas are about 300 years old, and the Euler–Maclaurin formula has a voluminous literature. We shall show that both of them can be interpreted as evaluations of integrals of certain interpolants to f which take the form of an algebraic polynomial plus a trigonometric polynomial. Amazingly, this seems not to have been noticed before. The interpolants can also be used for other applications besides quadrature.

Whereas the Euler–Maclaurin interpolant depends on derivatives of f at ± 1, the Gregory interpolant depends only on point values. This makes it readily usable in practical computation and a competitor with other schemes for interpolation of data at equally-spaced points. As is well known, polynomials are not suitable for equispaced interpolation in view of the Runge phenomenon [21], and it is a longstanding challenge to find alternative methods whose properties are as favorable as possible [20]. A particularly elegant choice is Floater–Hormann interpolation [8], where the interpolant is a rational function that is guaranteed to be pole-free on $[-1,1]$.

The ideas of Euler–Maclaurin and Gregory interpolation do not depend on the grid being equally spaced. For uneven grids, one may again do trigonometric interpolation after first making polynomial corrections at the endpoints (see [22], §9). Upon integrating, one gets analogues of Euler–Maclaurin and Gregory quadrature formulas for uneven grids, which will again be exponentially accurate if f is periodic.

2 Euler–Maclaurin interpolants

Let k be a nonnegative integer, and assume that $f \in C([-1,1])$ has derivatives through order $k-1$ at $x = \pm 1$. Let p_k be the unique polynomial of the form

$$p_k(x) = a_1 x + a_2 x^2 + \cdots + a_k x^k \tag{6}$$

that satisfies

$$p_k^{(j)}(1) - p_k^{(j)}(-1) = f^{(j)}(1) - f^{(j)}(-1), \quad 0 \le j \le k-1. \tag{7}$$

Existence and uniqueness are readily established by noting that the conditions imposed on the coefficients $\{a_j\}$ by (7) take the form of a triangular matrix with nonzero diagonal entries. Then $f - p_k$ has derivatives through order $k-1$ across $x = \pm 1$ when viewed as a periodic function on $[-1,1]$. Let $t_{k,n}$ denote its degree n trigonometric interpolant of the form (2), which will have pointwise accuracy $O(h^k)$ in $[-1,1]$, assuming f is smooth enough in the interior of the interval. (For $k=0$, it is sufficient for f to be Hölder or Lipschitz continuous, and for $k \ge 1$, it is sufficient for the kth derivative of f to be absolutely continuous. We shall not attempt to state the sharpest possible regularity assumptions.) We define the *order k Euler–Maclaurin interpolant* to f on (1) to be

$$f_{k,n} = p_k + t_{k,n}, \tag{8}$$

the sum of an algebraic polynomial of degree k and a trigonometric polynomial of degree n. We further define

$$E_{k,n}(f) = \int_{-1}^{1} f_{k,n}(x)dx. \tag{9}$$

For $k = 0$, we have $p_0 = 0$ and $f_{0,n} = t_{0,n}$, the trigonometric interpolant to f itself. Euler–Maclaurin interpolation is the same as trigonometric interpolation, and the pointwise accuracy is $O(1)$ because of the Gibbs phenomenon at $x = \pm 1$,

$$f_{0,n}(x) = f(x) + O(1).$$

(Here and in other similar statements we assume that f is sufficiently smooth.) The integral of $f_{0,n}$ gives the trapezoidal rule approximation to the integral of f,

$$E_{0,n}(f) = T_n(f) = I(f) + O(h^2).$$

One power of h is gained because the Gibbs oscillations are localized to a region of width $O(h)$, and the other because the leading-order term of the interpolation error is odd and integrates to zero.

For $k = 1$, we have $p_1(x) = a_1 x$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)].$$

Now $f - p_1$ is continuous across ± 1, so the amplitude of the Gibbs oscillations is reduced to $O(h)$,

$$f_{1,n}(x) = f(x) + O(h).$$

Since p_1 and $t_{1,n}$ both differ from their $k = 0$ analogues by odd functions, which integrate to zero, the integral of $f_{1,n}$ is the same as that of $f_{0,n}$,

$$E_{1,n}(f) = E_{0,n}(f).$$

For $k = 2$, we have $p_2(x) = a_1 x + a_2 x^2$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)], \quad a_2 = \tfrac{1}{4}[f'(1) - f'(-1)].$$

Now $f - p_2$ is C^1 across ± 1, and the pointwise accuracy improves to $O(h^2)$,

$$f_{2,n}(x) = f(x) + O(h^2).$$

The leading order interpolation error is odd again, integrating to zero, so again the quadrature error is better by two powers of h:

$$E_{2,n}(f) = I(f) + O(h^4).$$

Suppose we wish to work out an explicit formula for $E_{2,n}(f)$ in analogy to the formulas $E_{1,n}(f) = E_{0,n}(f) = T_n(f)$ above for the cases $k = 0, 1$. We can do this by calculating

$$E_{2,n}(f) = E_{2,n}(p_2) + E_{2,n}(f - p_2) = I(p_2) + T_n(f - p_2)$$

since the Euler–Maclaurin interpolant of p_2 is itself and the Euler–Maclaurin interpolant of $f - p_2$ is its trigonometric interpolant. This implies

$$E_{2,n}(f) = T_n(f) + I(p_2) - T_n(p_2),$$

and since

$$I(p_2) - T_n(p_2) = -\tfrac{1}{12}h^2(f'(1) - f'(-1)), \tag{10}$$

this yields

$$E_{2,n}(f) = T_n(f) - \tfrac{1}{12}h^2(f'(1) - f'(-1)). \tag{11}$$

To establish (10) we may first verify it by explicit computation in the case of a single trapezoid extending from $x = a$ to $x = b$. The result follows by concatenating n trapezoids and noting that since they all have the same width h, the contributions at interior boundaries cancel.

For $k = 3$, we have $p_3(x) = a_1x + a_2x^2 + a_3x^3$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)] - a_3, \quad a_2 = \tfrac{1}{4}[f'(1) - f'(-1)],$$
$$a_3 = \tfrac{1}{12}[f''(1) - f''(-1)]$$

and

$$f_{3,n}(x) = f(x) + O(h^3).$$

The differences from the case $k = 2$ are odd functions, integrating to zero, so we have

$$E_{3,n}(f) = E_{2,n}(f).$$

For $k = 4$, we have $p_4(x) = a_1x + \cdots + a_4x^4$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)] - a_3, \quad a_2 = \tfrac{1}{4}[f'(1) - f'(-1)] - 2a_4,$$
$$a_3 = \tfrac{1}{12}[f''(1) - f''(-1)], \quad a_4 = \tfrac{1}{48}[f'''(1) - f'''(-1)].$$

The function and quadrature approximations satisfy

$$f_{4,n}(x) = f(x) + O(h^4), \quad E_{4,n}(f) = I(f) + O(h^6).$$

For $k = 5$, we have $p_5(x) = a_1x + \cdots + a_5x^5$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)] - a_3 - a_5, \quad a_2 = \tfrac{1}{4}[f'(1) - f'(-1)] - 2a_4,$$
$$a_3 = \tfrac{1}{12}[f''(1) - f''(-1)] - \tfrac{10}{3}a_5, \quad a_4 = \tfrac{1}{48}[f'''(1) - f'''(-1)],$$
$$a_5 = \tfrac{1}{240}[f''''(1) - f'''(-1)].$$

The function and quadrature approximations satisfy

$$f_{5,n}(x) = f(x) + O(h^5), \quad E_{5,n}(f) = E_{4,n}(f).$$

The time has come to make the connection with the Euler–Maclaurin formula, an assertion about quadrature sums that was put forward independently around 1740 by Leonhard Euler in St. Petersburg and Colin Maclaurin in Edinburgh [7,16]. The formula can be written as an asymptotic series

$$I(f) \sim T_n(f) - h^2\frac{B_2}{2!}(f'(1) - f'(-1)) - h^4\frac{B_4}{4!}(f'''(1) - f'''(-1)) - \cdots, \tag{12}$$

where $\{B_k\}$ are the Bernoulli numbers ($B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots$). For any $k \geq 0$, if we form the *order k Euler–Maclaurin formula* by truncating (12) after terms involving derivatives of order up to $k-1$, then the error is $O(h^{k+1})$, or $O(h^{k+2})$ if k is even. In particular, the right-hand side of (11) may be recognized as the Euler–Maclaurin formula for $k = 2$ or $k = 3$.

A great deal is known about the Euler–Maclaurin formula, but we will not need much. To prove our basic theorem, all we need is the well-known property that the order k Euler–Maclaurin formula is exact when applied to a polynomial of degree k. Here is the theorem.

Theorem 1. *Given $k \geq 0$, let f have an absolutely continuous kth derivative on $[-1,1]$, and for each $n \geq 1$, let $f_{k,n}$ be its Euler–Maclaurin interpolant* (8)*. Then $f_{k,n}$ is an entire function that interpolates f on the grid* (1)*. (If $k = 0$, the interpolation condition at the endpoints is $f_{k,n}(\pm 1) = \frac{1}{2}(f(1)+f(-1))$.) Assuming $k \geq 1$, the interpolant satisfies*

$$f_{k,n}(x) = f(x) + O(h^k) \tag{13}$$

uniformly for $x \in [-1,1]$, and the integral $E_{k,n}(f) = \int_{-1}^{1} f_{k,n}(x)dx$ satisfies

$$E_{k,n}(f) = I(f) + O(h^{k+1}), \tag{14}$$

or $O(h^{k+2})$ if k is even and f has a continuous $(k+1)$st derivative on $[-1,1]$. Moreover, the number $E_{k,n}(f)$ is the same as the result of the order k Euler–Maclaurin formula applied to f.

Proof. The function $f_{k,n}$ is entire since it is the sum of an algebraic polynomial and a trigonometric polynomial. It interpolates f by construction. Concerning its integral, we compute

$$E_{k,n}(f) = E_{k,n}(p_k) + E_{k,n}(f - p_k) = I(p_k) + T_n(f - p_k) \tag{15}$$

since the order k Euler–Maclaurin interpolant of p_k is itself and the order k Euler–Maclaurin interpolant of $f - p_k$ is its trigonometric interpolant. On the other hand suppose we let $\tilde{E}_{k,n}(f)$ denote the number that results when the order k Euler–Maclaurin formula is applied to f. Then we have

$$\tilde{E}_{k,n}(f) = \tilde{E}_{k,n}(p_k) + \tilde{E}_{k,n}(f - p_k) = I(p_k) + T_n(f - p_k) \tag{16}$$

since the order k Euler–Maclaurin formula is exact for polynomials of degree k ([14], Corollary 3.3) and it reduces to the trapezoidal rule for a function like $f - p_k$ with continuous derivatives through order $k-1$ across ± 1. Comparing (15) and (16) establishes $\tilde{E}_{k,n}(f) = E_{k,n}(f)$, as claimed. Finally, concerning accuracy, we note that $f - f_{k,n}$ is the error in trigonometric interpolation of $f - p_k$, which can be regarded as a $(k-1)$-times differentiable periodic function on $[-1,1]$ with an absolutely continuous kth derivative. From standard theory of trigonometric interpolation, proved by integrating a Fourier series integral by parts $k+1$ times, the Fourier coefficients of $f - p_k$ decrease at the rate $O(n^{-k-1})$. By adding up a tail of such bounds with the aliasing formula for interpolation, it follows that if $k \geq 1$, then $\|f - f_{k,n}\|_\infty = O(h^k)$. (See [6], eq. (2.3.6) and [3], Theorem 2.1.) For the assertion (14) concerning the accuracy of the integral, see [4], Theorem 7.1.2. $\square$

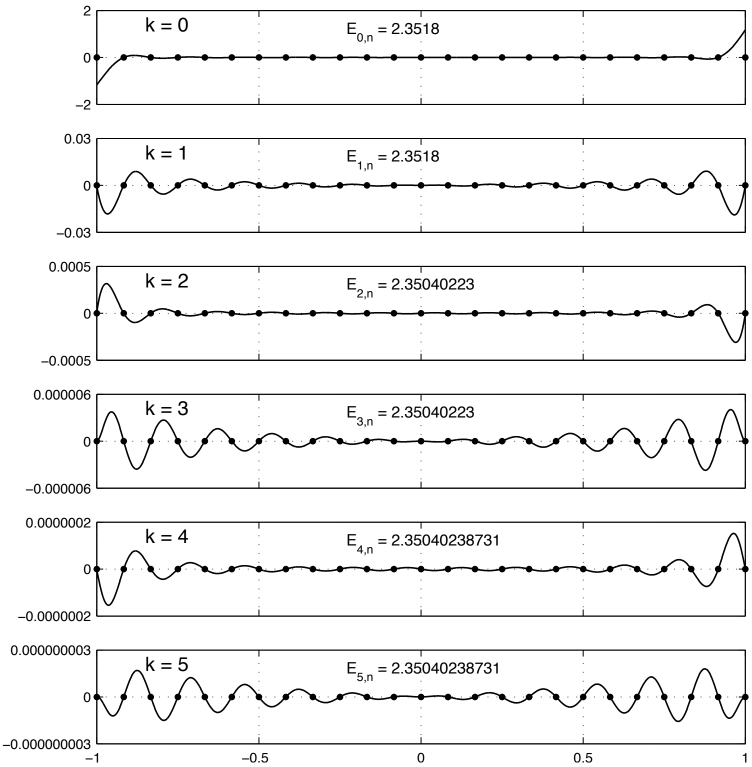

Fig. 1 Error functions $(f - f_{k,n})(x)$ for Euler–Maclaurin interpolants with $n = 24$ and $0 \leq k \leq 5$ for $f(x) = \exp(x)$. The first plot corresponds to trigonometric interpolation, while the others incorporate polynomial adjustments to make the functions C^0, C^1, C^2, C^3, and C^4, respectively, across ± 1. The exact value of the integral is ≈ 2.35040238760.

3 Examples of Euler–Maclaurin interpolation

Euler–Maclaurin interpolation is illustrated in Figure 1, which shows the error functions $(f - f_{k,n})(x)$ for $f(x) = \exp(x)$ with $n = 24$ and $k = 0, 1, \ldots, 5$. One sees that the interpolants are smooth and accurate, improving rapidly as k increases, and that the largest errors appear near the endpoints.

We would like to draw attention to a particular feature revealed in the figure. The function $\exp(x)$ is much smaller near $x = -1$ than near $x = 1$, so one might expect the oscillations to be much smaller on the left than on the right. Instead, they have roughly the same magnitude. This highlights the fact that despite their reliance on derivatives at the endpoints, Euler–Maclaurin interpolants are not local, but are derived from periodic trigonometric functions.

Figure 2 shows convergence as $n \to \infty$ for the same function f. The slopes match the predictions of Theorem 1, with approximation accuracy $O(h^k)$ and quadrature accuracy $O(h^{k+1})$, or $O(h^{k+2})$ when k is even.

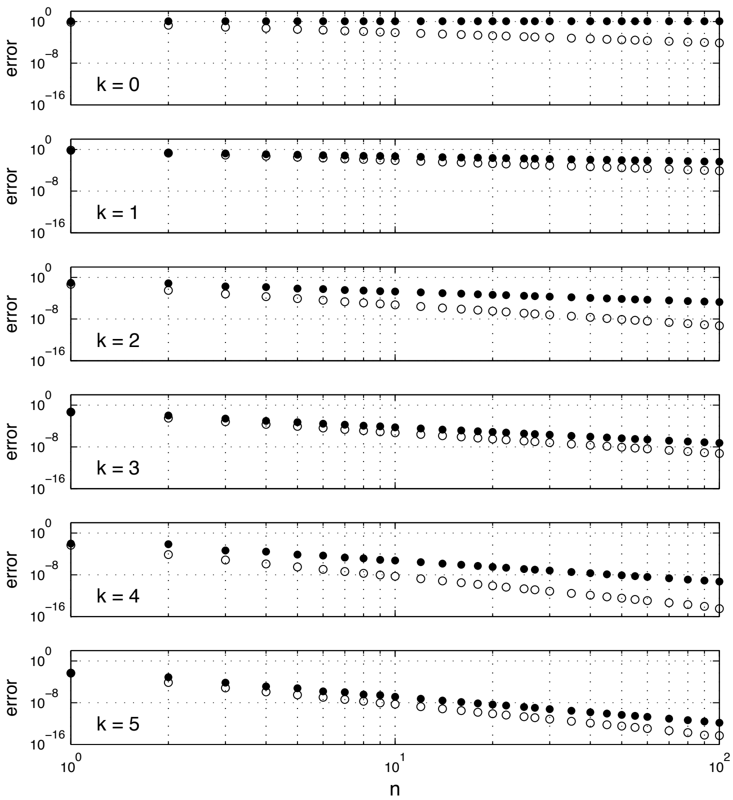

Fig. 2 Convergence of Euler–Maclaurin interpolants as $n \to \infty$ for the same function $f(x) = \exp(x)$ as in Figure 1. Solid dots show the ∞-norm of the approximation error, which scales as $O(h^k)$. Circles show the absolute value of the quadrature error, which scales as $O(h^{k+1})$, or $O(h^{k+2})$ if k is even.

For numerical computations involving Euler–Maclaurin interpolants, it is important to be able to compute effectively with trigonometric interpolants. This is best done with the use of the *barycentric interpolation formula,* which takes the form

$$t(x) = \sum_{j=0}^{n}{}' (-1)^j \cot(\frac{\pi}{2}(x-x_j)) f(x_j) \Big/ \sum_{j=0}^{n}{}' (-1)^j \cot(\frac{\pi}{2}(x-x_j)) \qquad (17)$$

when n is even and

$$t(x) = \sum_{j=0}^{n}{}' (-1)^j \csc(\frac{\pi}{2}(x-x_j)) f(x_j) \Big/ \sum_{j=0}^{n}{}' (-1)^j \csc(\frac{\pi}{2}(x-x_j)) \qquad (18)$$

when n is odd [2,12]. If $x = x_j$ exactly for some j, then instead of using (17) or (18), one sets $t(x) = f(x_j)$.

4 Gregory interpolants

If the derivatives of the order k Euler–Maclaurin quadrature formula are replaced by one-sided finite differences, one gets the Gregory quadrature formula [4,17], which is related to Newton–Gregory polynomial interpolation. We shall use the standard notation for the forward and backward difference operators on the grid,

$$\Delta f(-1) = f(-1+h) - f(-1), \quad \nabla f(1) = f(1) - f(1-h).$$

With this notation it can be shown that

$$\begin{aligned} I(f) \sim T_n(f) &- \frac{h}{12}[\nabla f(1) - \Delta f(-1)] - \frac{h}{24}[\nabla^2 f(1) - \Delta^2 f(-1)] \\ &- \frac{19h}{720}[\nabla^3 f(1) - \Delta^3 f(-1)] - \frac{3h}{160}[\nabla^4 f(1) - \Delta^4 f(-1)] - \cdots, \end{aligned} \qquad (19)$$

and if this series is truncated after the term of order $k-1$, the result is what we shall call the *order k Gregory formula,* which integrates smooth functions with accuracy $O(h^{k+1})$ [4,13].

Historically, Gregory and Newton were active long before Euler and Maclaurin, around 1670 [11]. (James Gregory was a brilliant young Scottish mathematician who was a significant influence on both Newton, in Cambridge, and Maclaurin, in Edinburgh.) Our development of a Gregory interpolant will closely follow that of the Euler–Maclaurin interpolant, with just a few necessary changes. One change is that p_k now depends on the grid, so it is relabeled $p_{k,n}$.

Let k be a nonnegative integer, and let $f \in C([-1,1])$ be arbitrary. We require the parameter n of (1) to satisfy $n \geq k$, so that there are at least $k-1$ interior grid points at which to impose conditions on $p_{k,n}$ associated with finite differences of $f - p_{k,n}$. The principle of finite difference approximations will be this: to compute the finite difference approximation to $f^{(j)}(-1)$ for the order k Gregory interpolant, we interpolate the data $f(-1), f(-1+h), \ldots, f(-1+(k-1)h)$ by a polynomial p of degree $k-1$, and then our discrete approximation is $p^{(j)}(-1)$. Similarly, to approximate $f^{(j)}(1)$, we interpolate $f(1-(k-1)h),\dots,f(1-h),f(1)$ and then evaluate $p^{(j)}(1)$. The resulting formulas can be expressed compactly in operator notation. If I, D, and E represent the identity, derivative, and shift operators defined by $If(x)=f(x)$, $Df(x)=f'(x)$, and $Ef(x)=f(x+h)$, then the formal identities $I+\Delta=E=\exp(hD)$ and $I-\nabla=E^{-1}=\exp(-hD)$ yield the formulas

$$hD = \quad \log(I+\Delta) = \Delta - \tfrac{1}{2}\Delta^2 + \tfrac{1}{3}\Delta^3 - \cdots \tag{20}$$

$$= -\log(I-\nabla) = \nabla + \tfrac{1}{2}\nabla^2 + \tfrac{1}{3}\nabla^3 + \cdots, \tag{21}$$

and similarly for powers $(hD)^j = [\log(I+\Delta)]^j = [-\log(I-\nabla)]^j$; see [13,19]. Applied to any polynomial, such formulas terminate as valid identities after a finite number of terms. From such calculations one obtains the asymptotic series at the left endpoint

$$\begin{aligned} hf'(-1) &\sim \Delta f(-1) - \tfrac{1}{2}\Delta^2 f(-1) + \tfrac{1}{3}\Delta^3 f(-1) - \cdots, \\ h^2 f''(-1) &\sim \Delta^2 f(-1) - \Delta^3 f(-1) + \tfrac{11}{12}\Delta^4 f(-1) - \cdots, \\ h^3 f'''(-1) &\sim \Delta^3 f(-1) - \tfrac{3}{2}\Delta^4 f(-1) + \tfrac{7}{4}\Delta^5 f(-1) - \cdots, \end{aligned}$$

and so on; the series at $x=1$ are the same except with backward differences and all signs positive. (For formulas and algorithms that apply also to nonequispaced points, see [9,10].) We define the order k forward difference approximation to $f^{(j)}(-1)$, denoted by $\Delta_k^{(j)} f(-1)$, to be the result obtained by truncating the jth series of this form after the term of degree $k-1$. The corresponding backward difference approximation to $f^{(j)}(1)$ is denoted by $\nabla_k^{(j)} f(1)$.

We now define the (k,n) *Gregory interpolant* of f, denoted by $g_{k,n}$. Let $p_{k,n}$ be the polynomial of the form (6) that satisfies conditions (7), except with all the derivatives in those conditions replaced by their one-sided finite difference approximations of order k. That is, $p_{k,n}$ is determined by the conditions

$$\nabla_k^{(j)} p_{k,n}(1) - \Delta_k^{(j)} p_{k,n}(-1) = \nabla_k^{(j)} f(1) - \Delta_k^{(j)} f(-1), \quad 0 \le j \le k-1. \tag{22}$$

Equation j of this set is a linear equation connecting $a_{j+1},\dots,a_k$ whose coefficient associated with a_{j+1} is nonzero, so existence and uniqueness of $p_{k,n}$ follow from triangular structure as before. Then $g_{k,n}$ is the function

$$g_{k,n} = p_{k,n} + t_{k,n}, \tag{23}$$

where $t_{k,n}$ is the degree n trigonometric interpolant to $f - p_{k,n}$ of the form (2). We further define

$$G_{k,n}(f) = \int_{-1}^{1} g_{k,n}(x)dx. \tag{24}$$

For $k=0$ and $k=1$, no derivatives of f appear in the definition of the Euler–Maclaurin interpolant $f_{k,n}$ of §2. Therefore the Gregory and Euler–Maclaurin interpolants are the same: $g_{0,n}=f_{0,n}$, $g_{1,n}=f_{1,n}$.

For $k=2$, we have $p_{k,n}(x) = a_1 x + a_2 x^2$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)], \quad a_2 = [\nabla f(1) - \varDelta f(-1)]/(4h - 2h^2).$$

The integral of this Gregory interpolant differs by $O(h^3)$ from the result of second-order Gregory formula, known as the *Durand rule.* (The Gregory interpolant reproduces x^2 exactly, whereas the Durand rule does not quite integrate it exactly.)

For $k=3$, we have $p_{k,n}(x) = a_1 x + a_2 x^2 + a_3 x^3$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)] - a_3,$$

$$a_2 = [(\nabla + \tfrac{1}{2}\nabla^2) f(1) \,-\, (\varDelta \,-\, \tfrac{1}{2}\varDelta^2) f(-1)]/4h,$$

$$a_3 = [\nabla^2 f(1) - \varDelta^2 f(-1)]/(6h^2 - 6h^3).$$

The integral of this Gregory interpolant is equal to the result of the third-order Gregory formula, known as the *Lacroix rule.* (The Gregory interpolant reproduces x^3 exactly, and the Lacroix rule integrates it exactly thanks to symmetry.)

For $k=4$, we have $p_{k,n}(x) = a_1 x + \cdots + a_4 x^4$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)] - a_3,$$

$$a_2 = [(\nabla + \tfrac{1}{2}\nabla^2 + \tfrac{1}{3}\nabla^3) f(1) - (\varDelta - \tfrac{1}{2}\varDelta^2 + \tfrac{1}{3}\varDelta^3) f(-1)]/4h + a_4/(8h^2 - 12h^5),$$

$$a_3 = [(\nabla^2 + \nabla^3) f(1) - (\varDelta^2 - \varDelta^3) f(-1)]/12h^2,$$

$$a_4 = [\nabla^3 f(1) - \varDelta^3 f(-1)]/(48h^3 - 72h^4).$$

The integral of this Gregory interpolant differs by $O(h^5)$ from the result of fourth-order Gregory formula, which makes a small error in integrating x^4.

For $k=5$, we have $p_{k,n}(x) = a_1 x + \cdots + a_5 x^5$ with

$$a_1 = \tfrac{1}{2}[f(1) - f(-1)] - a_3 - a_5,$$

$$a_2 = [(\nabla + \tfrac{1}{2}\nabla^2 + \tfrac{1}{3}\nabla^3 + \tfrac{1}{4}\nabla^2) f(1) - (\varDelta - \tfrac{1}{2}\varDelta^2 + \tfrac{1}{3}\varDelta^3 - \tfrac{1}{4}\varDelta^4) f(-1)]/4h - 2a_4,$$

$$a_3 = [(\nabla^2 + \nabla^3 + \tfrac{11}{12}\nabla^4) f(1) - (\varDelta^2 - \varDelta^3 + \tfrac{11}{12}\varDelta^4) f(-1)]/12h^2 - \tfrac{10}{3}(1 - 5h^3) a_5.$$

$$a_4 = [(\nabla^3 + \tfrac{3}{2}\nabla^4) f(1) - (\varDelta^3 - \tfrac{3}{2}\varDelta^4) f(-1)]/48h^3.$$

$$a_5 = [\nabla^4 f(1) - \varDelta^4 f(-1)]/(240h^4 - 480h^5).$$

The integral of this Gregory interpolant is equal to the result of the fifth-order Gregory formula, which integrates x^5 exactly thanks to symmetry.

A theorem for Gregory interpolants follows the same pattern as for Euler–Maclaurin interpolants, though with a restriction to odd values of k as the examples above explain.

Theorem 2. *Given $k \geq 0$, let f have an absolutely continuous kth derivative on $[-1,1]$, and for each $n \geq k$, let $g_{k,n}$ be its Gregory interpolant* (23). *Then $g_{k,n}$ is an entire function that interpolates f on the grid* (1). *(If $k=0$, the interpolation condition at the endpoints is $g_{k,n}(\pm 1) = \frac{1}{2}(f(1)+f(-1))$.) Assuming $k \geq 1$, the interpolant satisfies*

$$g_{k,n}(x) = f(x) + O(h^k) \tag{25}$$

uniformly for $x \in [-1,1]$, and the integral $G_{k,n}(f) = \int_{-1}^{1} g_{k,n}(x)dx$ satisfies

$$G_{k,n}(f) = I(f) + O(h^{k+1}). \tag{26}$$

For odd $k \geq 1$, the number $G_{k,n}(f)$ is the same as the result of the order k Gregory formula applied to f.

Proof. The function $g_{k,n}$ is entire since it is the sum of an algebraic polynomial and a trigonometric polynomial, and it interpolates f by construction. For its integral, we compute

$$G_{k,n}(f) = G_{k,n}(p_{k,n}) + G_{k,n}(f - p_{k,n}) = I(p_{k,n}) + T_n(f - p_{k,n}) \tag{27}$$

since the order k Gregory interpolant of $p_{k,n}$ is itself and the order k Gregory interpolant of $f - p_{k,n}$ is its trigonometric interpolant. On the other hand suppose we let $\tilde{G}_{k,n}(f)$ denote the number that results when the order k Gregory quadrature formula is applied to f. Then if k is odd, we have

$$\tilde{G}_{k,n}(f) = \tilde{G}_{k,n}(p_{k,n}) + \tilde{G}_{k,n}(f - p_{k,n}) = I(p_{k,n}) + T_n(f - p_{k,n}) \tag{28}$$

since the order k Gregory quadrature formula is exact for polynomials of degree k for k odd and it reduces to the trapezoidal rule for $f - p_{k,n}$ since $\Delta_k^{(j)}(f - p_{k,n})(-1) = \nabla_k^{(j)}(f - p_{k,n})(1)$ for $0 \leq j \leq k-1$ by (22). (The observation about exactness of Gregory formulas goes back at least as far as [1]; see also [4], Theorem 7.5.3.) Comparing (27) and (28) establishes $\tilde{G}_{k,n}(f) = G_{k,n}(f)$ for k odd.

To establish the accuracy claim (25), let us define $e_{k,n} = f - g_{k,n}$, the function whose trigonometric interpolant is $t_{k,n}$. We need to show that $\|e_{k,n} - t_{k,n}\| = O(h^k)$. Now $e_{k,n}$ is not smooth across the discontinuity at $x = \pm 1$, but by construction, its derivatives have small jumps there: $e'_{k,n}$ has a jump of size $O(h^{k-1})$, $e''_{k,n}$ has a jump of size $O(h^{k-2})$, and so on down to the condition that $e_{k,n}^{(k-1)}$ has a jump of size $O(h)$. Each of these discontinuities contributes $O(h^k)$ to $\|e_{k,n} - t_{k,n}\|$, adding up to a total of $O(h^k)$.

Finally we must establish the quadrature accuracy claim (26). From standard theory, we know that this estimate holds for $\tilde{G}_{k,n}(f)$, the result of Gregory quadrature; see [17] or Theorem 7.5.1 of [4]. For k odd, this is all we need since $G_{k,n}(f) = \tilde{G}_{k,n}(f)$. For k even, it is enough to establish $G_{k,n}(f) = \tilde{G}_{k,n}(f) + O(h^{k+1})$. This estimate holds because $G_{k,n}(f)$ and $\tilde{G}_{k,n}(f)$ differ in their treatments of the monomial x^k: $G_{k,n}$ integrates it exactly, whereas $\tilde{G}_{k,n}$ makes an error of size $O(h^{k+1})$, since it would make no error if it included one further term in (19), the kth difference term, which has size $O(h^{k+1})$. ◻

5 Examples of Gregory interpolation

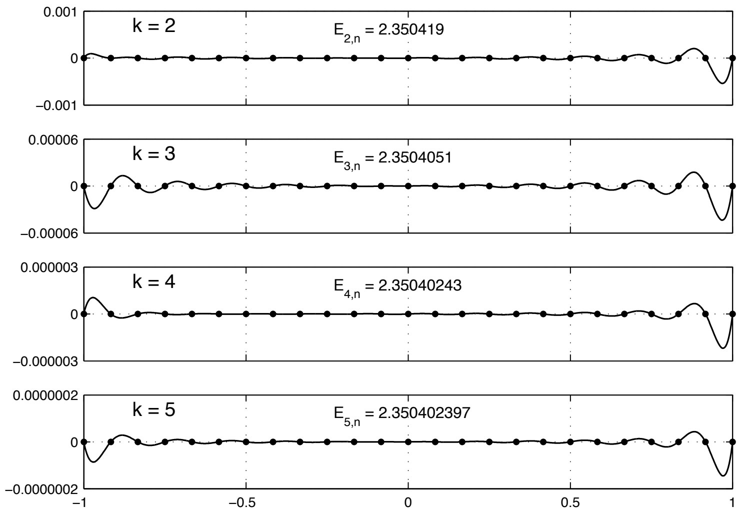

Fig. 3 Errors $f - g_{k,n}$ of Gregory interpolants with $n = 24$ and $k = 2, 3, 4, 5$ for $f(x) = \exp(x)$. Cases $k = 0$ and 1 are not shown since they would be identical to the corresponding cases of Figure 1. Note that the vertical scales are larger than in Figure 1, and that all the quadrature results differ. The exact value of the integral is ≈ 2.35040238760.

To illustrate Gregory interpolation, we follow the same format as in Figures 1 and 2, based on the function $f(x) = \exp(x)$. For $k = 0$ and $k = 1$, the Euler–Maclaurin and Gregory interpolants are identical, so Figures 3 and 4 show just $k = 3, 4, 5, 6$.

Comparing Figure 3 with Figure 1 and Figure 4 with Figure 2, we see that the amplitudes of the interpolation errors have increased by a factor of ten or more, though the orders of convergence remain the same. The increase can be explained by noting that the derivative terms in the Euler–Maclaurin formula may be a good deal smaller than the the errors introduced in discretizing them. For example, the coefficient $-19/720$ in (19) is the sum of $1/720$, from the Euler–Maclaurin formula (12), and $-20/720$, from the $\frac{1}{3}\Delta^3$ and $\frac{1}{3}\nabla^3$ terms in the discretizations (20) and (21) of $f'(-1)$ and $f'(1)$. Thus it is the discretization that contributes primarily to the error. One could vary the scheme to try to improve the balance, for example by using finite differences $\Delta_{k+1}^{(j)}(-1)$ and $\nabla_{k+1}^{(j)}(1)$ instead of $\Delta_k^{(j)}(-1)$ and $\nabla_k^{(j)}(1)$ in (22). Such strategies may have their uses in practice, but we shall not investigate them here as they break the connection with the Euler–Maclaurin and Gregory quadrature formulas.

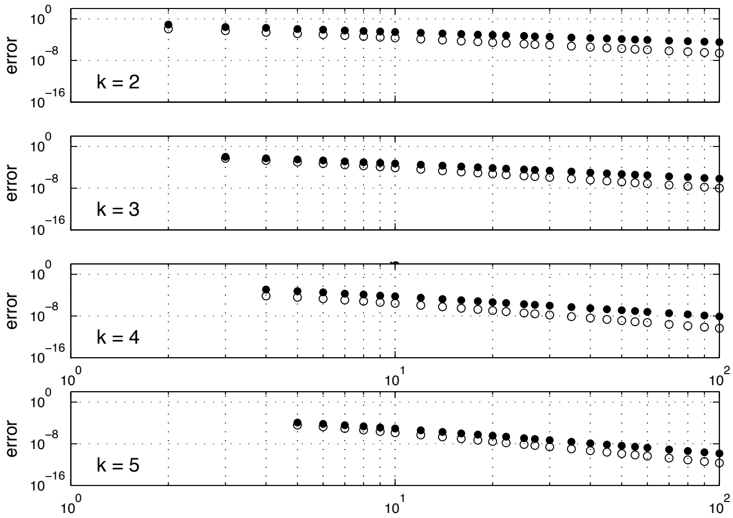

Fig. 4 Convergence of Gregory interpolants as $n \to \infty$ for $f(x) = \exp(x)$. Solid dots show the ∞-norm of the approximation error, which scales as $O(h^k)$. Circles show the absolute value of the quadrature error, which scales as $O(h^{k+1})$.

The experiments confirm that distinct values of k give distinct Gregory quadrature results, whereas for Euler–Maclaurin quadrature the numbers come in even/odd pairs.

6 Conclusion

The idea of interpolating samples of a function, then integrating the interpolant, is the central technique of numerical integration. When the interpolant is a polynomial, this is the basis of Gauss, Clenshaw–Curtis, and Newton–Cotes quadrature formulas. If one of these formulas is applied over multiple panels, the polynomial interpolant becomes a piecewise polynomial or spline. And so it is that the trapezoidal rule, for example, delivers the exact integral of—obviously—a piecewise linear interpolant. When the trapezoidal rule is improved to an Euler–Maclaurin or Gregory formula, the result is equal to the integral of a higher-order piecewise polynomial interpolant [15, 18, 23].

The trapezoidal rule also delivers the integral of a very different interpolant, a trigonometric polynomial, and it is this observation that explains its special accuracy for periodic functions or more generally functions that are smooth at the boundaries, and points the way to generalized formulas with similar accuracy for uneven grids. So far as we know, the present paper represents the first attempt to generalize this aspect of the trapezoidal rule to Euler–Maclaurin and Gregory formulas. Besides giving new insight into quadrature methods, our Euler–Maclaurin and Gregory interpolants may also prove useful in other applications involving smooth functions sampled at equispaced or approximately equispaced points.

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